The Beta Function

Gary Schurman, MBE, CFA

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Gabriele Veneziano, a research fellow at CERN (a European particle accelerator lab) in 1968, observed that many properties of the strong nuclear force are perfectly described by the beta function, an obscure formula devised for purely mathematical reasons two hundred years earlier by Leonhard Euler. This was, in effect, the birth of string theory.

The beta function, also called the Euler integral of the first kind, evolves from gamma function. The notation for the beta function is $\beta(a,b)$. Just as the gamma function for integers describes factorials, the beta function can define binomial coefficients. For example...

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{1}{n+1}\,\beta(a,b) \text{ ...where... } a = n-k+1 \text{ ...and... } b = k+1 \tag{1}$$

Note that the binomial coefficient is the number of ways of picking k unordered outcomes from n possibilities (i.e. order is not important).

The Beta Function Equation

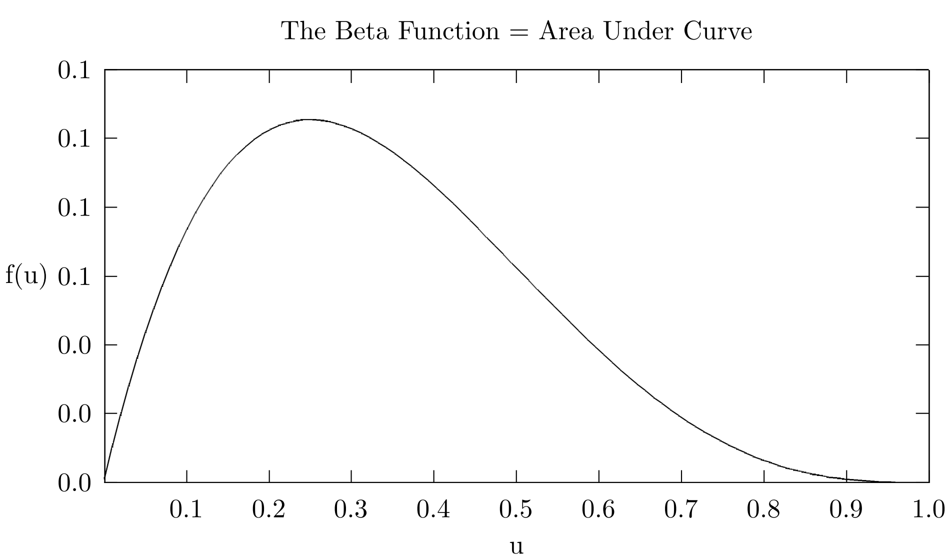
In the white paper **The Gamma Function** we defined the gamma function $\Gamma(\alpha)$ via the following equation...

$$\Gamma(\alpha) = \int\limits_{u=0}^{u=\infty} u^{\alpha-1}\,\mathrm{Exp}\left\{-u\right\}\delta u \text{ ...where... } \{\alpha\,|\,\alpha\in\mathbb{R},\alpha>0\} \tag{2}$$

The beta function $\beta(\alpha,\lambda)$, which is related to the gamma function above, is defined via the following equation...

$$\beta(\alpha,\lambda) = \int\limits_{u=0}^{u=1} u^{\alpha-1}\,(1-u)^{\lambda-1}\delta u \text{ ...where... } \{\alpha\,|\,\alpha\in\mathbb{R},\alpha>0\} \text{ ...and... } \{\lambda\,|\,\lambda\in\mathbb{R},\lambda>0\} \tag{3}$$

An example the beta function of $f(u,\alpha=2,\lambda=4)$ is the area under the following curve...



Note that we will have problems at the beta integral boundaries zero and one when α and/or λ is less than one. This statement in equation form is...

$$\lim_{u\to 0} u^{\alpha-1} = \infty \text{ ...when... } \alpha < 1 \;\; ; \;\; \lim_{u\to 1} (1-u)^{\lambda-1} \text{ ...when... } \lambda < 1 \tag{4}$$

For this reason many textbooks define the domain of the beta function to be $0 \le u \le 1$ when α and/or λ is less than one.

The Solution To The Beta Function Integral

Using Equation (2) above the equation for the product of two gamma functions of the variables α and λ, respectively, is...

$$\Gamma(\alpha)\,\Gamma(\lambda) = \int\limits_{u=0}^{u=\infty} u^{\alpha-1}\,\mathrm{Exp}\bigg\{-u\bigg\}\,\delta u \int\limits_{v=0}^{v=\infty} v^{\lambda-1}\,\mathrm{Exp}\bigg\{-v\bigg\}\,\delta v = \int\limits_{u=0}^{u=\infty}\int\limits_{v=0}^{v=\infty} u^{\alpha-1}\,v^{\lambda-1}\,\mathrm{Exp}\bigg\{-(u+v)\bigg\}\,\delta v\,\delta u \tag{5}$$

We will make the following definitions...

$$u = xy \text{ ...where... } \frac{\delta u}{\delta x} = y \text{ ...and... } \frac{\delta u}{\delta y} = x \text{ ...such that... } \delta u = y\,\delta x + x\,\delta y \tag{6}$$

And...

$$v = x\,(1-y) \text{ ...where... } \frac{\delta v}{\delta x} = 1-y \text{ ...and... } \frac{\delta v}{\delta y} = -x \text{ ...such that... } \delta y = (1-y)\,\delta x - x\,\delta y \tag{7}$$

Using the definitions in Equations (6) and (7) above the equation for the product of δu and δv is...

$$\delta u\,\delta v = y\,(1-y)\,\delta x^2 + x\,y\,\delta x\,\delta y + x\,(1-y)\,\delta x\,\delta y - x^2\delta y^2 \tag{8}$$

The solution to Equation (8) above is...

$$\delta u\,\delta v = x\,\delta x\,\delta y \text{ ...given that... } \lim_{x\to 0}\delta x^2 = 0 \text{ ...and... } \lim_{y\to 0}\delta y^2 = 0 \tag{9}$$

If we perform a change of variables from u and v to x and y then our next task is to redefine the bounds of integration for our new equation. Per Equation (5) above the domain of u is zero to infinity. The values of x and y that are consistent with this range are...

$$u = x\,y \text{ ...so... } x \in \{0,\infty\}, y \in \{0,\infty\} \text{ or } x \in \{0,-\infty\}, y \in \{0,-\infty\} \text{ ...such that... } \mathrm{Min}\ u = 0, \mathrm{Max}\ u = \infty \tag{10}$$

Per Equation (5) above the domain of v is zero to infinity. The values of x and y that are consistent with this range are...

$$v = x\,(1-y) \text{ ...so... } x \in \{0,\infty\}, y \in \{0,1\} \text{ ...such that... } \mathrm{Min}\ v = 0, \mathrm{Max}\ v = \infty \tag{11}$$

The bounds of integration that are consistent with both Equations (10) and (11) is...

$$x \in \{0,\infty\}, y \in \{0,1\} \text{ ...such that... } \mathrm{Min}\ u = 0, \mathrm{Max}\ u = \infty \text{ ...and... } \mathrm{Min}\ v = 0, \mathrm{Max}\ v = \infty \tag{12}$$

Using Appendix Equation (16) below the solution to Equation (5) above is...

$$\Gamma(\alpha)\,\Gamma(\lambda) = \int\limits_{u=0}^{u=\infty}\int\limits_{v=0}^{v=\infty} u^{\alpha-1}\,v^{\lambda-1}\,\mathrm{Exp}\bigg\{-(u+v)\bigg\}\,\delta v\,\delta u = \Gamma(\alpha+\lambda)\,\beta(\alpha,\lambda) \tag{13}$$

Using Equation (13) above the solution to beta function integral Equation (3) above is...

$$\beta(\alpha,\lambda) = \frac{\Gamma(\alpha)\,\Gamma(\lambda)}{\Gamma(\alpha+\lambda)} \tag{14}$$

Appendix

A. The gamma function of the sum of two variables a and b is...

$$\Gamma(a+b) = \int\limits_{u=0}^{u=\infty} u^{a+b-1}\, \mathrm{Exp}\left\{ -u \right\} \delta u \quad \text{...where...} \quad \{a+b \,|\, a+b \in \mathbb{R}, a+b > 0\} \tag{15}$$

B. Using Equations (6), (7), (9), (12) and (15) above the solution to Equation (5) above is...

$$\begin{aligned}
\Gamma(\alpha)\,\Gamma(\lambda) &= \int\limits_{u=0}^{u=\infty} \int\limits_{v=0}^{v=\infty} u^{\alpha-1}\, v^{\lambda-1}\, \mathrm{Exp}\left\{ -(u+v) \right\} \delta v\, \delta u \\
&= \int\limits_{x=0}^{x=\infty} \int\limits_{y=0}^{y=1} (x\,y)^{\alpha-1}\, x^{\lambda-1}\, (1-\; y)^{\lambda-1}\, \mathrm{Exp}\left\{ -(x\,y + x - x\,y) \right\} x\, \delta x\, \delta y \\
&= \int\limits_{x=0}^{x=\infty} \int\limits_{y=0}^{y=1} x^{\alpha+\lambda-2}\, y^{\alpha-1}\, (1-y)^{\lambda-1}\, \mathrm{Exp}\left\{ -x \right\} x\, \delta x\, \delta y \\
&= \int\limits_{x=0}^{x=\infty} \int\limits_{y=0}^{y=1} x^{\alpha+\lambda-1}\, \mathrm{Exp}\left\{ -x \right\} y^{\alpha-1}\, (1-y)^{\lambda-1}\, \delta x\, \delta y \\
&= \int\limits_{x=0}^{x=\infty} x^{\alpha+\lambda-1}\, \mathrm{Exp}\left\{ -x \right\} \delta x \int\limits_{y=0}^{y=1} y^{\alpha-1}\, (1-y)^{\lambda-1}\, \delta y \\
&= \Gamma(\alpha+\lambda)\, \beta(\alpha,\lambda)
\end{aligned} \tag{16}$$